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Stability of Embedded Non-Foster Metamaterials with Potentially Unstable Circuit Parameters

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***Abstract* – Recent advances have motivated renewed interest in the development of wideband metamaterials using non-Foster circuits. In contrast to passive metamaterials, the presence of active circuits in non-Foster metamaterials requires consideration of stability issues. Stability arguments for non-Foster metamaterials are often predicated on analysis of lumped-element representations of individual unit cells comprising the metamaterial. The present work considers the use of such potentially unstable non-Foster unit cells to form a stable system in the limit, under certain embedding constraints. Stability of an embedded active metamaterial is first considered. Then, non-Foster elements are introduced into a lumped-element transmission line model of a unit cell. Combining the two analyses, conditions are presented for stable systems in the limit, with straightforward extension to multidimensional non-Foster metamaterials.**

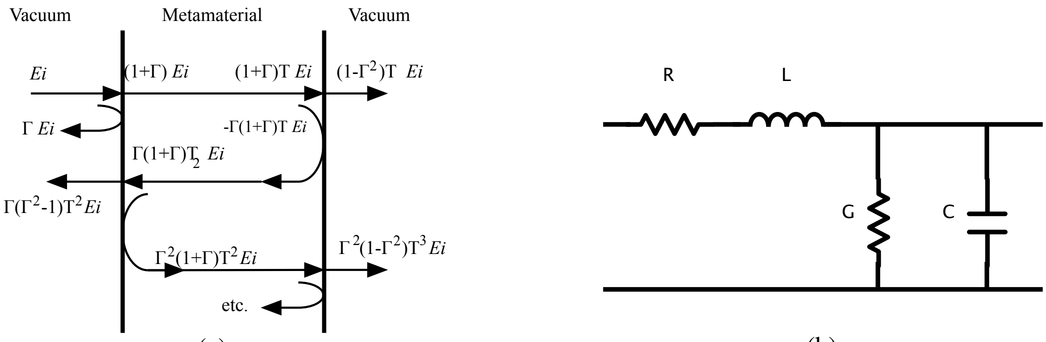
I. INTRODUCTION

A number of investigators have recently presented promising results in the development of non-Foster metamaterials such as wideband artificial magnetic conductors [1], wideband composite metamaterial and metasurfaces [2], wideband metamaterial structures [3, 4], and measurements of wideband epsilon-near-zero metamaterials with gain [5]. However, the active devices inherent in non-Foster metamaterials present the potential for instability [6, 7]. One approach to the study of stability in non-Foster unit cells is to consider the time-domain response of lumped-element equivalent circuits [7, 8, 9]. The basic notion is that the common $e^{-t/RC}$ impulse response of a simple RC circuit diverges if either R or C is negative. As pointed out by some researchers, the embedding network and terminations have significant effect on overall stability, and many classical stability measures may become inappropriate or misleading [6, 8]. Furthermore, investigators have considered Clausius-Mossotti and Routh-Hurwitz approaches in extending results to the many unit cells comprising a metamaterial [7, 8]. In the present work, an equivalent lumped-element transmission line model is used to bridge the gap between a number of these prior notions of circuit models, stability, embedding, termination, and effective medium.

In the following, a two-step process is used to analyze the stability of a slab of metamaterial embedded in a surrounding medium. In the first step, general stability conditions are given for an embedded metamaterial slab using reflection and transmission parameters of the material [10, 11]. In the second step, a lumped-element transmission line section model of a unit cell is used to consider the stability of active non-Foster metamaterials in terms of circuit parameters. The resulting stability limits can provide insight to the development of stable non-Foster metamaterial formulations when embedding effects are included.

II. STABILITY ANALYSIS OF AN EMBEDDED METAMATERIAL SLAB

For simplicity, first consider the stability of an active metamaterial embedded within an external medium as illustrated in Fig. 1(a) [10]. Although vacuum is shown, it will become apparent that the development is suited for any embedding medium, and that the following approach closely resembles related work on parameter extraction [10, 11]. In Fig. 1(a), an incoming electromagnetic plane wave E_i impinges on the left face of the metamaterial slab, where the time factor $e^{j\omega t}$ is suppressed. The reflected component at the left face is ΓE_i, where $\Gamma = (\eta - \eta_\circ) / (\eta + \eta_\circ)$, η is the impedance of the material, and $\eta_\circ = 377\,\Omega$ is the vacuum impedance. The transmitted component is then $(1 + \Gamma)E_i$. After propagating through the metamaterial slab of thickness d, the wave



(a) (b)

Fig. 1: Models. (a) Model of metamaterial embedded in a medium (here a vacuum). (b) Lumped-element model of a transmission section representing a unit cell of a metamaterial.

arriving at the right face is then $(1+\Gamma)T\,E_i$, where $T = e^{-jk(\omega)d}$ for a metamaterial with frequency-dependent wavenumber $k(\omega)$. Similarly, the wave reflects again, with the first reflected component at the right face becoming $-\Gamma(1+\Gamma)T\,E_i$, and the first transmitted component at the right face becoming $(1-\Gamma^2)T\,E_i$. As previously noted for problem of parameter extraction [11], the infinite reflections result in the following S-parameters for the for the system of Fig. 1(a):

$$S_{11} = \Gamma\left(1+T^2\left(\Gamma^2-1\right)\sum_{n=0}^{\infty}\left(\Gamma^2T^2\right)^n\right) = \frac{\Gamma\left(1-T^2\right)}{1-\Gamma^2T^2} \;; \qquad \text{if } |\Gamma\,T|<1\,, \tag{1}$$

$$S_{21} = T\left(1-\Gamma^2\right)\sum_{n=0}^{\infty}\left(\Gamma^2T^2\right)^n = \frac{T\left(1-\Gamma^2\right)}{1-\Gamma^2T^2} \;; \qquad \text{if } |\Gamma\,T|<1\,, \tag{2}$$

where the infinite sums converge for $|\Gamma\,T|<1$. For present purposes, it is important to note that the overall system can have stable gain as long as $|\Gamma\,T|<1$, even when $|S_{21}|>1$, and even when using gain media with $|T|>1$. The result follows along lines similar to laser cavity gain computation.

For the ensuing analysis, the simple lumped-element transmission line section of Fig. 1(b) is used to model a unit cell in a metamaterial [12]. For a transmission line composed of such sections with distributed resistance R (Ω/m), inductance L (H/m), conductance G (S/m), and capacitance C (F/m), the characteristic impedance is $Z=\sqrt{(R+j\omega L)/(G+j\omega C)}$ and propagation constant is $\gamma = jk(\omega) = \sqrt{(R+j\omega L)(G+j\omega C)}$. For lossless passive metamaterials, $R=G=0$, Z is real, γ is imaginary, k is real, and $|T| = |e^{-jk(\omega)d}| = |e^{-\gamma d}| = 1$ indicating no gain or loss in the transmission line. For a non-Foster metamaterial using the model of Fig. 1(b), some number of the parameters R, G, L, and C will be negative, and the behavior becomes fairly complicated. Nevertheless, the stability condition $|\Gamma\,T|<1$ becomes:

$$|\Gamma T| = \left|\frac{Z-Z_\circ}{Z+Z_\circ}e^{-\gamma d}\right| = \left|\frac{\sqrt{\frac{R+j\omega L}{G+j\omega C}}-Z_\circ}{\sqrt{\frac{R+j\omega L}{G+j\omega C}}+Z_\circ}e^{-\sqrt{(R+j\omega L)(G+j\omega C)}\,d}\right| < 1\,, \tag{3}$$

with proper choice of roots throughout, as noted in [10, 13, 14]. The more general result in (3) is fairly complicated, so for the purpose of illustration consider the simplified case of $|R| \ll |\omega L|$ and $|G| \ll |\omega C|$, where:

$$Z = \sqrt{L/C}\sqrt{(1-jR/(\omega L))/((1-jG/(\omega C))} \approx \sqrt{L/C}\;e^{-jR/(2\omega L)}e^{jG/(2\omega C)} \tag{4}$$

$$\gamma = j\omega\sqrt{LC}\sqrt{(1-jR/(\omega L))((1-jG/(\omega C))} \approx j\omega\sqrt{LC}\;e^{-jR/(2\omega L)}e^{-jG/(2\omega C)}\;. \tag{5}$$

Then, the gain/loss becomes $|T| \approx e^{-\mathrm{Re}\{\gamma d\}}$, where $\mathrm{Re}\{\gamma d\} \approx \omega\sqrt{LC}\;\left[R/(2\omega L)+G/(2\omega C)\right]d$. To further simplify, let $Z_\circ = \eta_\circ = \sqrt{L/C}$, and then $\Gamma = (Z-Z_\circ)/(Z+Z_\circ) \approx (Z_\circ e^{-jR/(2\omega L)}e^{jG/(2\omega C)} - Z_\circ)/(Z_\circ e^{-jR/(2\omega L)}e^{jG/(2\omega C)}+Z_\circ)$, and $\Gamma \approx jG/(4\omega C) - jR/(4\omega L)$ for $|R| \ll |\omega L|$ and $|G| \ll |\omega C|$. Then, the convergence condition (2) may simplify to $|\Gamma T| \approx \left|\left[jG/(4\omega C)-jR/(4\omega L)\right]\left[e^{-\omega\sqrt{LC}\;[R/(2\omega L)+G/(2\omega C)]d}\right]\right|$. Continuing, $|\Gamma T| \approx \left|\left[jG/(4\omega C)-jR/(4\omega L)\right]\left[1-\omega\sqrt{LC}\;\left[R/(2\omega L)+G/(2\omega C)\right]d\right]\right|$ when $\omega R d\sqrt{L/C}$ and

$\omega G d\sqrt{C/L}$ are sufficiently small. Finally, note that $|\Gamma T| \approx \left| G/(4\omega C)\left[1{-}\omega\sqrt{LC}\; Gd/(2\omega C)\right]\right|$ and the metamaterial is stable so long as $\left| G/(4\omega C)\left[1{-}\omega\sqrt{LC}\; Gd/(2\omega C)\right]\right| < 1$, with final simplification $R = 0$ for exposition.

Although this example has been overly simplified, it serves to illustrate the proposed approach to stability conditions, so long as the criterion $|\Gamma T| < 1$ is met, and even if the gain through the system is greater than unity. The less restrictive, though somewhat complicated, condition in (3) has the potential to offer a wider range of conditions for stable non-Foster metamaterials having various combinations of negative values of R, G, C, or L.

III. CONCLUSION

Combining the stability analysis of an embedded active metamaterial slab with lumped element line-section models of unit cells, metamaterial stability conditions are presented in the limit. The models can be extended to multidimensional non-Foster metamaterials. Less complicated stability conditions were offered under simplifying assumptions. The approach can offer insight for the development and the analysis of non-Foster metamaterials.

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