I. INTRODUCTION

A Dubins path is the planar path of minimum length and bounded curvature that connects two specified endpoints with specified approach angles [1]. Because of this optimality property, and because there is a simple, geometric procedure for constructing them, Dubins paths are often used for guidance of constant-speed, planar vehicles [2] [3] [4]. For such a vehicle, often referred to as a “Dubins car,” a time-optimal path consists of maximum rate turns and straight line segments. However, a Dubins path that is constructed using the true maximum turn rate cannot be tracked in the presence of disturbances, because feedback commands may exceed the turn rate limit.

If there is sufficient control authority, and the disturbances are perfectly known (whether steady [5], [6], [7] or unsteady [8]), then minimum time trajectories to the goal state can be planned that account for these disturbances explicitly. In general, though, disturbances are unknown and unavailable for planning and control purposes. One approach to dealing with uncertain disturbances is to dispense with planning altogether and to use feedback control to drive the system toward a desired end goal. In [9], for example, an optimal control law is presented that drives the vehicle to a target set (with a free final course angle) in the presence of a stochastically varying wind. In some applications, however, such as directional sensing or vehicle recovery operations, attaining a desired position with a prescribed course angle may be important.

This note describes how the Dubins path planning method can be modified to construct sub-optimal paths that remain feasible in the presence of a bounded, unsteady disturbance. The path is planned using an artificially reduced “maximum” turn rate that is a function of the (known) upper bound on the disturbance magnitude. Though the resulting path is longer, it could be tracked by the vehicle if the disturbance were known. For unknown disturbances, the reserve control authority enables a path following algorithm to force convergence to the desired path.

II. PROBLEM FORMULATION

Consider a vehicle that moves in an inertial, horizontal plane at a constant forward speed $V_r$ and in some direction $\psi$ relative to a reference frame $F_T$ fixed in the plane. For the moment, assume that there are no disturbances acting on the vehicle. The vehicle’s position is given by the pair $(x, y)$. The input is the turn rate $u$, which is symmetrically bounded. A turn at maximum rate corresponds to a circular path of minimum radius $R_0$. The input constraint may be expressed in terms of this minimum turn radius and the vehicle’s speed:

$$|u| \leq \frac{V_r}{R_0}$$

The equations of motion, which define the standard Dubins car model, are

$$
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{\psi}(t)
\end{pmatrix} =
\begin{pmatrix}
V_r \cos \psi(t) \\
V_r \sin \psi(t) \\
u(t)
\end{pmatrix}
$$

(1)
A. Disturbances

Suppose there is a time-varying velocity disturbance of unknown magnitude \( V_\delta(t) \in [0, V_{\delta\text{max}}] \) acting on the vehicle in some unknown direction \( \psi_\delta(t) \in [0, 2\pi) \), where \( V_\delta(t) \) and \( \psi_\delta(t) \) are continuously differentiable and \( V_{\delta\text{max}} > 0 \) is a known upper bound on the disturbance magnitude. For example, this velocity disturbance might represent a wind gust acting on an unmanned aerial vehicle (UAV). To ensure that a feasible path exists for any desired final state, it is assumed that \( V_{\delta\text{max}} < V_r \) where \( V_r \) is now the flow-relative vehicle speed. The parameter \( \epsilon \) is defined to be the ratio of the maximum disturbance magnitude to the vehicle’s flow-relative speed:

\[
\epsilon = \frac{V_{\delta\text{max}}}{V_r} < 1
\]

Incorporating this disturbance, the system \([1]\) becomes

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{\psi}(t)
\end{pmatrix} =
\begin{pmatrix}
V_r \cos \psi(t) + V_\delta(t) \cos \psi_\delta(t) \\
V_r \sin \psi(t) + V_\delta(t) \sin \psi_\delta(t) \\
u(t)
\end{pmatrix}
\]

One may construct a path that is feasible for the system \([3]\) by constructing a Dubins path for the system \([1]\) using a “feasible turn radius” \( R_0' \) which is larger than \( R_0 \), but which affords the vehicle more turn rate authority to compensate for the disturbance. As shown in the Appendix, given \( V_\delta(t) \leq V_{\delta\text{max}} \) and \( \psi_\delta(t) \), a Dubins path planned using the turn radius

\[
R_0' = R_0(1 + \epsilon)^2
\]

is feasible for the system \([3]\), meaning there exists a control \( u^*(t) \) for which the vehicle perfectly follows the desired path. While \( u^*(t) \) exists, determining this input requires exact knowledge of the disturbance. Advanced sensors may enable such feedforward disturbance rejection, but it is more common that only the vehicle’s state (position and course angle) is directly measured. In this case, one may use feedback control to diminish the effect of disturbances.

The result \([4]\) is closely related to Lemma 6 in \([10]\) wherein the maximum turn rate required to maintain a fixed circular orbit in a known time-invariant flowfield is derived. This maximum turn rate corresponds to the feasible turn radius \( R_0' \) as defined here. However, the present work demonstrates that \( R_0' \) holds for a more general case, where only an upper bound on the disturbance magnitude is known and the flowfield is assumed to be time-varying, continuously differentiable, and possibly with a known steady component as discussed in the following section.

B. Steady, Uniform Wind with Disturbances

If there is a known, steady, uniform wind in addition to the random disturbances, one may account for this wind explicitly. Several methods have been developed to plan time-optimal paths in known, steady winds \([5]\), \([6]\), \([7]\). The resulting optimal path always consists of trochoids and straight segments in the fixed, inertial frame \( \mathcal{F}_T \). Equivalently, it consists of circular arcs and straight segments in an air-relative frame \( \mathcal{F}_A \) that moves with the mean wind velocity. As shown in the Appendix, the result \([4]\) extends to the case where there is a known, steady, uniform wind under the additional restriction that \( V_w + V_{\delta\text{max}} < V_r \), where \( V_w \) is the steady, mean wind speed.

III. TRADEOFF BETWEEN PATH LENGTH AND PATH FEASIBILITY

One may assume without loss of generality that the vehicle begins at the state \( q_0 = [0 \ 0 \ 0]^T \), so that the path planning problem is completely defined by the desired final state \( q = [x \ y \ \psi]^T \). As discussed in \([2]\) the Dubins path to the generic state \( q \) can be described using symbols for the sense of the turns (“L” for left and “R” for right circular segments) and the symbol “S” for straight segments. The complete set of candidate optimal paths is \{LRL, RLR, LSR, LSL, RSL, RSR\}. Since optimal paths may be parameterized by the final position \((x, y)\) and the final course angle \( \psi \), one may construct a diagram, for a given value of \( \psi \), which identifies regions of the \((x, y)\) plane in which minimum time paths from the origin to a given final point, with the final course angle \( \psi \), are of a given type; see \([11]\) \([12]\). One may thus associate a “partition” \( P_\psi \) with each final course angle \( \psi \in [0, \pi] \), recognizing that the partition \( P_{-\psi} \) is the reflection of \( P_\psi \) about the \( x \) axis. Figures \([1(a)\) \([1(b)\) and \([1(c)\) illustrate three such partitions for final course angles \( \psi = 0, 2\pi/3 \) and \( \pi \), respectively. The unit of length for the axes is the minimum turn radius \( R_0 \), so that points in the plane are given in normalized position coordinates \((\bar{x}, \bar{y}) = (x/R_0, y/R_0)\).
For a given final course angle $\psi$ and a final position $(\bar{x}, \bar{y})$, one may compare the length of the optimal Dubins path (planned with $R_0$) to that of the sub-optimal feasible Dubins path (planned with $R'_0$) by taking the ratio of the path lengths. Referring to the length of each of these paths as $d_{\text{optimal}}(\bar{x}, \bar{y}, \psi)$ and $d_{\text{feasible}}(\bar{x}, \bar{y}, \psi, \epsilon)$ respectively, the ratio

$$T(\bar{x}, \bar{y}, \psi, \epsilon) = \frac{d_{\text{feasible}}}{d_{\text{optimal}}}$$

gives a measure of the sub-optimality of the feasible path. A large value of $T$ indicates that allowing for a disturbance by using the planning radius $R'_0$ significantly increases the path length.

The “tradespace” $T(\bar{x}, \bar{y}, \psi, \epsilon)$ is illustrated for several pairs $(\psi, \epsilon)$. One can generate contours of the ratio $T(\bar{x}, \bar{y}, \psi, \epsilon)$ by synthesizing the optimal paths using $R_0$ and $R'_0$ for a grid of points around the origin. The examples in Figures 2(a), 2(b) and 2(c) illustrate this tradespace for $\epsilon = 0.25$ and for final course angles $\psi = 2\pi/3$ and $\pi$, respectively. In each figure the “thermal scale” representing the ratio of normalized path lengths is truncated at 5 for clarity and the contour lines are drawn in intervals of 0.2.

Figures 2(a), 2(b) and 2(c) indicate that, for points far from the origin, the relative cost of using a larger turn radius to plan the Dubins path is small. However, both figures have regions, especially within 4 turn radii of the origin, where there is a dramatic increase in relative path length. This issue can be understood by referring to the partitions in Figure 1. Consider two paths that are planned to the same point $(x, y)$, one using $R_0$ as the planning radius and another using $R'_0 > R_0$. The path planned using $R_0$ maps to the point $\frac{1}{R_0}(x, y)$ in the partition, and the path planned with $R'_0$ maps to the point $\frac{1}{R'_0}(x, y)$, which is nearer the origin. If these two points map to different regions within the partition then the qualitative type of each path will be different, possibly leading to a large path length ratio $T(\bar{x}, \bar{y}, \psi, \epsilon)$. 

FIG. 1: Partitions of the configuration space for final course angles $\psi = 0, 2\pi/3$ and $\pi$, respectively.

FIG. 2: Contour maps of the ratio of optimal path lengths planned using $R'_0$ and $R_0$. 
IV. PATH FOLLOWING PERFORMANCE

As mentioned in Section II.A to follow a Dubins path planned with the feasible turn radius \(R'_0\) requires perfect knowledge of the disturbances. Since accurate ambient flow measurements are typically unavailable, a feedback control system based on position measurements is typically used instead. Well-designed feedback control can enhance robustness to model uncertainty and disturbances, provided there is sufficient control authority to execute the control commands. The point of this note is to suggest a path planning strategy that produces suboptimal paths which can still be followed in the presence of unknown disturbances. Specifically, it is suggested that Dubins paths planned using the feasible turn radius \(R'_0\) will be “robust” to unknown disturbances, in the sense that these paths can be closely followed in spite of these disturbances. In this section, this hypothesis is investigated numerically.

In this numerical investigation, a UAV is modeled as a Dubins car moving at air-relative speed \(V_r = 20\) m/s and with a turn rate limit \(|u| \leq 0.3\) rad/s. For closed-loop path following, the lookahead-based guidance law for regularly parameterized paths described in [13] is used. The course commands generated by the algorithm in [13] are tracked with a bang-bang turn rate controller; the control parameters are chosen heuristically to give reasonable performance over a wide range of wind conditions and turn radii. This path following algorithm is illustrated in Figures 3(a) and 3(b) for two sample endpoints in a steady, uniform wind blowing in the positive \(x\) direction (with \(\epsilon = 0.25\)). In these figures, two paths are planned to each state \(q\); one path is planned using the true minimum turn radius \(R_0\) (dashed line) and the other using \(R'_0\) (dashed-dotted line). The UAV’s motion is simulated (solid line) using the guidance law from [13] to follow a given path. It seems clear from these figures that path following performance is improved for the paths planned using the feasible turn radius \(R'_0\), rather than \(R_0\). For both examples, the vehicle more closely follows the path planned using \(R'_0\), resulting in greater accuracy for the endpoint and the final course angle. The tradeoff, of course, is the increase in path length. To study the effect of varying the turn radius used in path planning on the path following performance, a Monte Carlo simulation study was conducted.

![Comparison of closed-loop path following performance](image)

**FIG. 3:** Comparison of closed-loop path following performance (solid lines) for paths planned to the state \(q\) using turn radii \(R_0\) (dashed line) and \(R'_0\) (dashed-dotted line).

A Monte Carlo simulation was used to compare the path following performance for various choices of the turn radius \(R_{\text{plan}}\) used in path planning. A number of measures could be used to assess path following performance for these different choices of planning radius. One might measure overall tracking performance, for example, using the integral squared offset from the desired path. Because the ultimate aim is to attain a desired final position and course angle, however, the measure used here was the final cross-track error – the distance between the vehicle and the endpoint of the desired path at the instant the vehicle crosses an imaginary “finish line” orthogonal to the the desired path at its endpoint; see Figures 3(a) and 3(b).

Both steady and unsteady disturbances were considered in these simulations. Unsteady disturbances were simulated using the Dryden turbulence model with power spectral densities that are representative of those an aircraft would experience in flight; see [14] and [15]. The resulting root-mean-square value of the gust magnitude is denoted \(\sigma\). In this simulation, \(\epsilon = \sigma/V_r\) and a saturation limit was imposed such that \(V_\delta(t) \leq \epsilon V_r\). Steady disturbances (constant winds) for a given bound \(\epsilon\) were simulated by taking \(V_\delta = \epsilon V_r\) and selecting a random, constant wind direction \(\psi_\delta\) from a uniform distribution for each sample in the simulation.

Monte Carlo simulations were conducted for a range of disturbance bounds \(\epsilon \in \{0.25, 0.50, 0.75, 0.95\}\) and planning radii \(R_{\text{plan}} \in \{1.0, 1.1, 1.2, \cdots, 4.0\}\), where \(R_{\text{plan}}\) is the turn radius used for planning, normalized by \(R_0\).
Each sample endpoint was randomly chosen from a square domain (of width and height $8R_0$) centered at the origin. Rather than sample over the entire range of final course angles, the final angle was selected randomly from the set $\psi_f \in \{0, 2\pi/3, \pi\}$, corresponding to the examples in Section III. The simulation was conducted for both the steady and unsteady disturbance case. For each sample in the simulation, a path was planned to the sampled endpoint using the given, nondimensional planning radius $\bar{R}_{\text{plan}}$, and the UAV was simulated following this path, using the guidance law from [13], in the presence of a disturbance of bounded magnitude. The final, nondimensional cross-track error was then recorded. For each disturbance bound $\epsilon$ and planning radius $\bar{R}_{\text{plan}}$, simulations for 900 samples (final states) were run in order to generate a single data point: the mean, final cross-track error for the given pair ($\epsilon, \bar{R}_{\text{plan}}$); see Figures 4(a) and 4(b).

![Diagram](image)

**FIG. 4**: Normalized mean, final cross-track error versus $\bar{R}_{\text{plan}}$ from Monte Carlo simulation.

In the results shown in Figures 4(a) and 4(b), the data points with $\bar{R}_{\text{plan}}$ closest to $R_0^f/R_0 = (1 + \epsilon)^2$ for a given $\epsilon$ are emphasized with a solid marker. In both cases, the trend is for the mean, final cross-track error to decrease as the planning radius increases until a critical value of the planning radius is reached (the “knee” in the curve). At this critical value, the mean, final cross-track error approaches a lower bound and no further performance improvements are gained by using larger values of $\bar{R}_{\text{plan}}$. This critical value of $\bar{R}_{\text{plan}}$ is of interest because it indicates the smallest turn radius (and hence the shortest path) for which the mean, final cross-track error is near its minimum value. For the steady disturbance case, the paths planned using the feasible turn radius $R_0^f$ are very near the “knee” in the curve. In the unsteady case, however, the paths planned using $R_0^f$ are conservative; this result is not surprising since the derivation of $R_0^f$ in the Appendix shows that the case of constant disturbance magnitude and direction ($\dot{V}_\delta = 0$ and $\psi_0 = 0$) is the limiting case, requiring the largest turn radius for feasibility.

**V. CONCLUSION**

The ability to compensate for disturbances when following a Dubins path can be improved by artificially increasing the turn radius used in path planning. Consider a Dubins car with a nominal minimum turn radius $R_0$ that is moving with a flow-relative speed $V$, subject to unknown disturbances of bounded magnitude $\epsilon V$, with $\epsilon < 1$. A Dubins path that is planned using the turn radius $R_0' = R_0(1 + \epsilon)^2$ is feasible, in the sense that it could be exactly followed if the disturbance were known. In practice, one typically uses feedback control to compensate for an unknown disturbance. In this case, Monte Carlo simulations suggest that these feasible Dubins paths (planned using the turn radius $R_0'$) minimize the mean, final cross-track error when the path is followed using a standard guidance algorithm. Furthermore, the simulation results indicate that there is no appreciable benefit in planning paths with a turn radius $R_{\text{plan}} > R_0'$; the planning radius $R_{\text{plan}} = R_0'$ marks the “knee” in the cross-track error curve for a steady disturbance. Simulation results also suggest that choosing $R_{\text{plan}} > R_0'$ may be conservative for unsteady disturbances.

The cost of using the (sub-optimal) feasible turn radius $R_0'$ for path planning is an increase in the nominal path length and travel time. This tradeoff can be illustrated, for a given final course angle, using a tradespace plot that indicates the relative path lengths of the ideal and feasible Dubins paths. These tradespace plots can be helpful in selecting a desired end state because they highlight regions where a given final state will result in a large increase in the relative path length.
Appendix

The following derivations are adapted from [16].

**Proposition 1.** Given $V_\delta(t) \leq V_{\delta_{\text{max}}}$ and $\psi_\delta(t)$, a circular turn of radius

$$R'_0 = R_0(1 + \epsilon)^2$$

is feasible for the system (3).

**Proof.** The radius of curvature $R(t)$ of the path defined by the trajectory $(x(t), y(t))$ of equations (3) in $\mathcal{F}_x$ is

$$R(t) = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}$$

(5)

Noting that

$$\ddot{x} = -V_r \sin(\psi)u + \dot{V}_\delta \cos(\psi_\delta) - V_\delta \sin(\psi_\delta) \dot{\psi}_\delta$$

$$\ddot{y} = V_r \cos(\psi)u + V_\delta \sin(\psi_\delta) + V_\delta \cos(\psi_\delta) \dot{\psi}_\delta$$

and substituting into (5)

$$R(t) = \frac{\varphi_3}{|\varphi_1 + \varphi_2|}$$

where

$$\varphi_1 = V_r^2 + V_\delta V_\delta \cos(\psi - \psi_\delta)$$

$$\varphi_2 = V_\delta^2 \dot{\psi}_\delta - V_r V_\delta \sin(\psi - \psi_\delta) + V_r V_\delta \cos(\psi - \psi_\delta) \dot{\psi}_\delta$$

$$\varphi_3 = (V_r^2 + V_\delta^2 + 2V_r V_\delta \cos(\psi - \psi_\delta))^3/2$$

Observe that $\varphi_1 > 0$ and $\varphi_3 > 0$, given condition (2). Using the triangle and Cauchy-Schwarz inequalities, recalling the input constraint, and rearranging gives

$$\frac{\varphi_3}{R(t)\varphi_1} - \frac{\varphi_2}{\varphi_1} \leq |u| \leq \frac{V_r}{R_0}$$

It follows that the range of attainable curvature is

$$R(t) \geq \frac{R_0\varphi_3}{V_r\varphi_1 + R_0|\varphi_2|}$$

(6)

Assume that there exists an upper bound on the right hand side (RHS) term of (6) and denote this as upper bound $R'_0$. Then it follows that the inertial turn radius $R(t)$ can always be made equal to or greater than $R'_0$ by using a feasible control input. Hence, $R'_0$ is the “feasible turn radius” – the smallest possible turn radius that can be feasibly tracked.

To find this upper bound $R'_0$ proceed by maximizing the RHS of (6). Note that the RHS will be maximized if $\varphi_2 = 0$. Since the terms $\dot{V}_\delta$ and $\dot{\psi}_\delta$ do not appear elsewhere, $\varphi_2$ can be equal to zero independently of $\varphi_1$ and $\varphi_3$. This occurs, for example, when the disturbance is constant in magnitude and direction ($\dot{V}_\delta = 0$ and $\dot{\psi}_\delta = 0$), but it also occurs in other cases where the disturbances take certain “pathological” forms. With $\varphi_2 = 0$ the RHS of (6) becomes

$$\frac{R_0\varphi_3}{V_r\varphi_1} = \frac{R_0(V_r^2 + 2V_r V_\delta \cos(\psi - \psi_\delta) + V_\delta^2)^{3/2}}{V_r(V_r^2 + V_\delta^2 \cos(\psi - \psi_\delta))}$$

(7)

For any $V_\delta$, the maximum of (7) occurs when $\cos(\psi - \psi_\delta) = 1$, this corresponds to a situation where the disturbance is always aligned with the vehicle’s motion. Then (7) simplifies to

$$\frac{R_0(V_r + V_\delta)^2}{V_r^2}$$
which is maximum when the disturbance magnitude is largest, \( V_\delta = V_{\delta_{\text{max}}} \). Defining

\[
R'_0 = \max \left( \frac{R_0 \varphi_3}{V_t \varphi_1 + R_0 |\varphi_2|} \right)
= R_0 \frac{(V_r + V_{\delta_{\text{max}}})^2}{V_t^2}
= R_0 (1 + \epsilon)^2
\]

yields the result.

**Corollary 2.** For \( V_{\delta_{\text{max}}} < V_r \), a straight path is feasible for the system (3).

Corollary 2 follows immediately from Proposition 1 because \( R \to \infty \) for a straight path.

**Proposition 3.** A Dubins path constructed for the system (7) using the turn radius \( R'_0 \) is feasible for the system (3).

**Proof.** In Proposition 1 and Corollary 2 it was shown that (inertial) circular segments of radius \( R'_0 \) and (inertial) straight segments are feasible. Since Dubins paths comprise circular and straight segments, it follows that a Dubins path constructed for (7) using the turn radius \( R'_0 \) is feasible for the system (3).

The above results are extended to the case of steady uniform winds by adding constant wind terms to the system (3) and considering motion in an air-relative frame. By adding a steady uniform wind with magnitude \( V_w \) in the direction \( \psi_w \), the system (3) becomes

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{\psi}(t)
\end{pmatrix}
=
\begin{pmatrix}
V_r \cos \psi(t) + V_\delta(t) \cos \psi_\delta(t) + V_w \cos \psi_w \\
V_r \sin \psi(t) + V_\delta(t) \sin \psi_\delta(t) + V_w \sin \psi_w \\
u(t)
\end{pmatrix}
\]  

(8)

To ensure a feasible solution \( V_w + V_{\delta_{\text{max}}} < V_r \) is required.

**Corollary 4.** For any \( \psi_\delta(t) \) and \( V_\delta(t) \leq V_{\delta_{\text{max}}} \) with constant \( V_w \) such that \( V_w + V_{\delta_{\text{max}}} < V_r \), there exists a suitably large turn radius \( R'_0 \) in the moving, air-relative reference frame \( \mathcal{F}_A \), that is feasible for the system (8). The feasible turn radius is \( R'_0 = R_0 (1 + \epsilon)^2 \) with \( \epsilon \) defined in (2).

**Proof.** Define the air-relative reference frame \( \mathcal{F}_A \) as an inertial frame that is convected in the direction of the ambient wind \( \psi_w \), at speed \( V_w \). In the frame \( \mathcal{F}_A \), the steady wind terms vanish, and the system becomes

\[
\begin{pmatrix}
\dot{x}_A(t) \\
\dot{y}_A(t) \\
\dot{\psi}(t)
\end{pmatrix}
=
\begin{pmatrix}
V_r \cos \psi(t) + V_\delta(t) \cos \psi_\delta(t) \\
V_r \sin \psi(t) + V_\delta(t) \sin \psi_\delta(t) \\
u(t)
\end{pmatrix}
\]  

(9)

System (9) is of the same form as system (3), however \( x_A \) and \( y_A \) are air-relative positions. Note that in \( \mathcal{F}_A \), the minimum turn radius is still \( R_0 \), and using the same approach described in the proof of Proposition 1 the result is obtained that the feasible turn radius in the moving, air-relative frame is \( R'_0 = R_0 (1 + \epsilon)^2 \).

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**References**


